Two Trivial Problems and an Impossible One

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Acknowledgements:

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- Special thanks to the organizers of SUMRY 2017, to Stefan Steinerberger for introducing me to the problem, and to Nathan Fox and Kevin O'Bryant for valuable insight and examples.

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Question

If we can extend the algorithm $\mathcal{A}$ so that it can accept as inputs non-standard integers n and k, what information does this give us about the family S_n?

First Example:

Definition (Hofstader, "Gödel, Escher, Bach")

The *Hofstader Q-sequence* is defined by
$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$ and initial conditions
$Q(1) = 1$ and $Q(2) = 1$.

- The first few terms are
  $1, 1, 2, 3, 3, 4, 5, 5, 6, 6, 6, 8, 8, 8, 10, 9, 10, 11, 11 \ldots$
- Open question whether this sequence is infinite or not.

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Definition (Fox 2018)

Define the sequence Q_r by the recurrence relation $Q_r(n) = Q_r(n - Q_r(n-1)) + Q_r(n - Q_r(n-2))$ and initial conditions $Q_r(1) = 1, Q_r(2) = 2, \ldots Q_r(r) = r$.

Non-standard Inputs:

- Clearly, there is an algorithm such that $\mathcal{A}(r, k) = Q_r(1), Q_r(2), Q_r(3), \ldots Q_r(k)$.
- I claim that this algorithm can actually be extended to allow non-standard inputs.

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- By Łoś's Theorem, we can construct a set ${}^*\mathbb{N}$, called the hyper-naturals, such that:
  - ${}^*\mathbb{N}$ contains the naturals
  - ${}^*\mathbb{N}$ contains an element N larger than any standard natural
  - We can lift subsets, functions, etc. on $\mathbb{N}$ to corresponding objects on ${}^*\mathbb{N}$, preserving first-order truth predicates.

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- E.g. $\forall x \in \mathbb{N}, x > 1 \Rightarrow x^2 > x$ implies $N^2 > N$, and a similar manner we can prove $N^2 > CN$ for any standard natural C.

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- Recall that $Q_N(n) = Q_N(n - Q_N(n-1)) + Q_N(n - Q_N(n-2))$.
- We can keep computing in this way until we hit the $(N+29)$-nd term.

$$\begin{aligned} Q_N =& 1, 2, 3, \dots N-1, N, 3, N+1, N+2, 5, N+3, 6, 7, N+4, \\ & N+6, 10, 8, N+6, N+10, 12, N+7, 14, N+12, 11, \\ & N+11, N+15, 16, 13, 17, 15, N+14, 20, 20, 2N+8. \end{aligned}$$

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$$\begin{aligned} Q_N(N+29) &= Q_N(N+29 - Q_N(N+28)) + Q_N(N+29 - Q_N(N+27)) \\ &= Q_N(N+29-2N-8) + Q_N(N+29-20) \\ &= Q_N(21-N) + Q_N(N+9) \text{ ☹} \end{aligned}$$

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- This can be phrased in a first-order way, and so we conclude that for all *naturals* N sufficiently large, the sequence Q_N has $N + 28$ terms!
- The bad news is that this isn't exciting: there is a completely elementary proof of an even stronger result in *A New Approach to the Hofstadter Q-Recurrence*, Fox 2018.

Second Example:

Definition

A *Sidon set* is a set $S \subset \mathbb{N}$ such that $\forall w, x, y, z \in S$, $w + x = y + z$ if and only if $\{w, x\} = \{y, z\}$.

An (A, B)-*form Sidon set* is a set $S \subset \mathbb{N}$ such $\forall w, x, y, z \in S$, $Aw + Bx = Ay + Bz$ if and only if $\{w, x\} = \{y, z\}$.

The *greedy* (A, B)-*form Sidon sequence* $S_{A,B}$ is the sequence starting with 0, such that each subsequent term is the next smallest term such that the sequence is an (A, B)-form Sidon set.

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$$S_{1,1} = 0, 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97 \ldots$$
$$S_{1,2} = 0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69 \ldots$$
$$S_{1,3} = 0, 1, 2, 9, 10, 11, 18, 19, 20, 81, 82, 83 \ldots$$

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- Because the extension to hyper-naturals preserves first-order statements, each term t in $S_{1,N}$ is the smallest such that for all $w, x, y, z \in S_{1,N} \cap [1, t]$, $w + xN = y + zN$ if and only if $\{w, x\} = \{y, z\}$.

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- Thus, at each step, we need to check if $t = x + (y - z)N$ or $t = x + (y - z)/N$ for $x, y, z \in S_{1,N} \cap [1, t - 1]$. This can be done recursively.

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$$\begin{aligned} S_{1,N} \cap [1, N^2 + N] =& 0, 1, 2, \ldots N - 1, \\ & N^2, N^2 + 1, \ldots N^2 + N - 1 \end{aligned}$$

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$$S_{1,N} = [1, 2N^2 + N] = 0, 1, 2, \ldots N - 1,$$
$$N^2, N^2 + 1, \ldots N^2 + N - 1,$$
$$2N^2, 2N^2 + 1, 2N^2 + 2, \ldots 2N^2 + N - 1$$

Non-standard Algorithm:

- In this recursive fashion, we can prove that

$$x \in S_{1,N} \Leftrightarrow \exists T \in {}^*\mathbb{N} \text{ s.t. } x = \sum_{l=0}^{T} a_l N^{2l},\ 0 \leq a_l < N.$$

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- Unfortunately, it is a theorem in the folklore (due to Kevin O'Bryant) that this is true for all $N \geq 2$, and this is again proved by elementary means.

Third Example:

Definition

An *Ulam sequence* is an increasing sequence $U(a, b)$ of integers defined by

- $u_0 = a$, $u_1 = b$, and
- u_k (for $k > 1$) is the smallest integer that can be written as the sum of two distinct smaller terms u_m, u_n in exactly one way.

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Examples:

- $U(1, 2) : 1, 2, 3, 4, 6, 8, 11, 13, 16, 18 \ldots$
- $U(1, 3) : 1, 3, 4, 5, 6, 8, 10, 12, 17, 21 \ldots$
- $U(2, 3) : 2, 3, 5, 7, 8, 9, 13, 14, 18, 19 \ldots$

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- Introduced in 1964 by Ulam, who wanted to understand their growth properties.
- Despite their apparent simplicity, almost nothing is known about Ulam sequences.

Rigidity of the $U(1, n)$ Sequences:

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$U(1,5)$:	$\boxed{1}$,	$\boxed{5,6,7,8,9,10}$,	$\boxed{12}$,	$\boxed{20}$,	$\boxed{22,23,24}$,		$\boxed{26}$ . . .
$U(1,6)$:	$\boxed{1}$,	$\boxed{6,7,8,9,10,11,12}$,	$\boxed{14}$,	$\boxed{24}$,	$\boxed{26,27,28,29}$,		$\boxed{31}$ . . .

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$U(1,n):$	$\boxed{1}$,	$\boxed{n,\ldots 2n}$,	$\boxed{2n+2}$,	$\boxed{4n}$,	$\boxed{4n+2,\ldots 5n-1}$,	$\boxed{5n+1}\ldots$

The Rigidity Conjecture:

Conjecture

There exists a positive integer N and integer coefficients m_i, p_i, k_i, r_i such that for all $n \geq N$,

$$U(1, n) = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i].$$

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- This is very well supported numerically (more on that later).
- Note that the coefficients don't depend on n, and can be calculated using any two consecutive Ulam sequences.
- Effectively, the conjecture says that once you have seen two (sufficiently large) Ulam sequences $U(1, n)$, you have seen them all.

Next Best Result:

Theorem (Weak Rigidity Theorem)

There exist integer coefficients m_i, p_i, k_i, r_i such that for every $C > 0$, there exists a positive integer N such that for all $n \geq N$,

$$U(1, n) \cap [1, Cn] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, Cn].$$

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- We shall prove this by making use of the machinery we have developed.

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$$1, N, N+1, N+2, \dots 2N-1, 2N, \cancel{2N+1}, 2N+2, 4N, \dots$$

- To make this formal, argue by induction on C and i.
- We thus construct m_i, p_i, k_i, r_i such that

$$U(1,N) \cap [1, CN] = \bigsqcup_{i \in \mathbb{N}} [m_i N + p_i, k_i N + r_i] \cap [1, CN].$$

Passing to the Hyper-Naturals:

- Fix a standard natural $C > 0$.
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- In fact, we produce an algorithm $\mathcal{A}$ capable of constructing these coefficients up to C!

Consequences:

- We have therefore proved over the hyper-naturals that for all sufficiently large N,

$$U(1, N) \cap [1, CN] = \bigsqcup_{i \in \mathbb{N}} [m_i N + p_i, k_i N + r_i] \cap [1, CN].$$

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- Also the first example where the theorem is not known independently.
- The proof is vaguely non-constructive, but we can make the result completely constructive.

Growth Rate of Coefficients:

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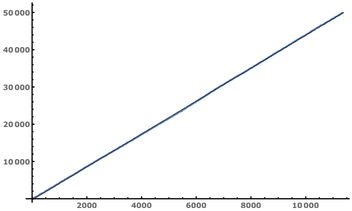
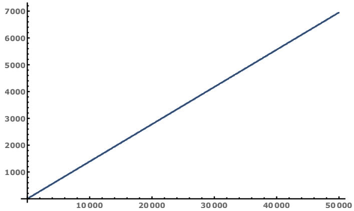
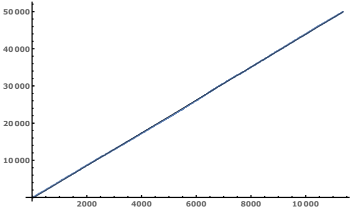


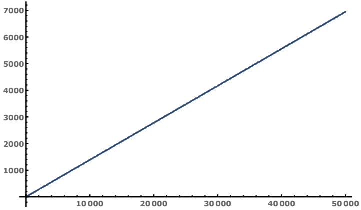
i vs. m_i



m_i vs. p_i



i vs. k_i



k_i vs. r_i

Growth Rate of Coefficients:

- In particular, for all i such that $k_i \leq 50000$, we find that

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- This is useful, because we can use this statement about the growth rate to make the weak rigidity theorem effective.

Effective Estimates:

Theorem

Suppose that for some positive integer M,

$$U(1, N_0) \cap [1, k_M N_0 + r_M + 1] = \bigsqcup_{i=1}^{M} [m_i N_0 + p_i, k_i N_0 + r_i]$$

where for some $B, \epsilon > 0$, $|p_i - m_i B|, |r_i - k_i B| < \epsilon$, and $N_0 > 4(1+\epsilon) - B$. Then for all $N > N_0$,

$$U(1, N) \cap [1, k_M N + r_M + 1] = \bigsqcup_{i=1}^{M} [m_i N + p_i, k_i N + r_i] .$$

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- The proof proceeds by induction over M and N.

Effective Estimates:

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  1. Calculate coefficients m_i, p_i, k_i, r_i.
  2. Do a linear regression to fit the best value of B to the computed coefficients. Calculate the corresponding maximum error ϵ.
  3. Compute $N_0' = \lceil 4(1+\epsilon) - B \rceil$.
  4. Use the coefficients m_i, p_i, k_i, r_i to predict the first CN terms of $U(1, N_0'), U(1, N_0' \pm 1) \ldots$.
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  5. Halt once you find the smallest N_0 such that $U(1, n)$ matches the prediction for all $N_0 \leq n \leq \max\{N_0, N_0'\}$.
- Using this, we prove that for all $n \geq 4$,

$$U(1, n) \cap [1, 50000n] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, 50000n].$$

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- What other families of integer sequences can we find that fit this general paradigm? (Known examples are variations on the listed ones.)
- Are there any general theorems that we can prove about integer sequences coming from an algorithm extendable to non-standard inputs?
- If we can prove some restrictions on the growth rate of the sequences, does this tells us something, like it does for the Ulam sequence?
- Does there exist any $\epsilon > 0$ such that there are integer coefficients m_i, p_i, k_i, r_i so that for any $C > 0$, there is an $N > 0$ such that for all $n \geq N$,

$$U(1, n) \cap [1, Cn^{1+\epsilon}] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, Cn^{1+\epsilon}]?$$