Rigidity in the Ulam Sequence

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Acknowledgements:

- Joint work with Joshua Hinman, Borys Kuca, and Alexander Schlesinger. (*The Unreasonable Rigidity of Ulam Sequences* and *Rigidity of Ulam Sets and Sequences*.)

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- Special thanks to the organizers of SUMRY 2017, and to Stefan Steinerberger for introducing me to the problem.

Introduction:

Definition

An *Ulam sequence* is an increasing sequence $U(a, b)$ of integers defined by

- $u_0 = a$, $u_1 = b$, and
- u_k (for $k > 1$) is the smallest integer that can be written as the sum of two smaller terms u_m, u_n in exactly one way.

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Examples:

- $U(1, 2) : 1, 2, 3, 4, 6, 8, 11, 13, 16, 18 \ldots$
- $U(1, 3) : 1, 3, 4, 5, 6, 8, 10, 12, 17, 21 \ldots$
- $U(2, 3) : 2, 3, 5, 7, 8, 9, 13, 14, 18, 19 \ldots$

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- Introduced in 1964 by Ulam, who wanted to understand their growth properties.
- Despite their apparent simplicity, almost nothing is known about Ulam sequences.

Open Problems: Order of Growth

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- Best known (general) bound: the Ulam sequence does not grow faster than exponentially fast.
- This is the trivial bound: the Ulam sequence doesn't grow faster than the Fibonacci sequence.
- Numerical evidence: the Ulam sequence grows linearly, regardless of initial conditions a, b.

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  - $a = 4$ (for $b \equiv 1 \mod 4$, proved by Cassaigne and Finch 1995)
  - $a = 5$, $b = 6$
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  - $a = 5$, $b = 6$
  - $a \geq 6$, and a or b is even.
- We prove $U(a, b)$ is periodic if (a, b) is on the following list.

$(4, 11)$	$(4, 19)$	$(6, 7)$	$(6, 11)$	$(7, 8)$	$(7, 10)$	$(7, 12)$
$(7, 16)$	$(7, 18)$	$(7, 20)$	$(8, 9)$	$(8, 11)$	$(9, 10)$	$(9, 14)$
$(9, 16)$	$(9, 20)$	$(10, 11)$	$(10, 13)$	$(10, 17)$	$(11, 12)$	$(11, 14)$
$(11, 16)$	$(11, 18)$	$(11, 20)$	$(12, 13)$	$(12, 17)$	$(13, 14)$	

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$U(1,2):$	1,	2,	3,	4,	6,	8,	11,	13,	16,	18,	26,	28 ...
$U(1,3):$	1,	3,	4,	5,	6,	8,	10,	12,	17,	21,	23,	28 ...
$U(1,4):$	1,	4,	5,	6,	7,	8,	10,	16,	18,	19,	21,	31 ...
$U(1,5):$	1,	5,	6,	7,	8,	9,	10,	12,	20,	22,	23,	24 ...
$U(1,6):$	1,	6,	7,	8,	9,	10,	11,	12,	14,	24,	26,	27 ...

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Are families of Ulam sequences rigid?

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\begin{array}{l|llllll}
U(1,2): & \boxed{1}, & \boxed{2,3,4}, & \boxed{6}, & 8, & 11, & 13\ldots \\
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U(1,6): & \boxed{1}, & \boxed{6,7,8,9,10,11,12}, & \boxed{14}, & \boxed{24}, & \boxed{26,27,28,29}, & \boxed{31}\ldots
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U(1,n): & \boxed{1}, & \boxed{n,\dots 2n}, & \boxed{2n+2}, & \boxed{4n}, & \boxed{4n+2,\dots 5n-1}, & \boxed{5n+1}\dots
\end{array}
$$

The Rigidity Conjecture:

Conjecture

There exists a positive integer N and integer coefficients m_i, p_i, k_i, r_i such that for all $n \geq N$,

$$U(1,n) = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i].$$

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In fact, we conjecture that we can take $N = 4$.

- This is very well supported numerically (more on that in a second).
- Note that the coefficients don't depend on n, and can be calculated using any two consecutive Ulam sequences.
- Effectively, the conjecture says that once you have seen two (sufficiently large) Ulam sequences $U(1, n)$, you have seen them all.

The Next Best Results:

- We prove unconditionally that there exist integer coefficients m_i, p_i, k_i, r_i such that for all $n \geq 4$,

$$U(1, n) \cap [1, 50000n] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, 50000n].$$

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- Where does this come from?

The Next Best Results:

Theorem (Weak Rigidity Theorem)

For every $C > 0$, there exists a positive integer N and integer coefficients m_i, p_i, k_i, r_i such that for all $n \geq N$,

$$U(1, n) \cap [1, Cn] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, Cn].$$

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- Note that the required N could grow indefinitely as C increases, so we cannot conclude the full rigidity conjecture from this.
- The proof is very unexpected, as it makes an appeal to model theory.

Model Theory:

- Note that once we fix a constant $C > 0$ and coefficients m_i, p_i, k_i, r_i, the statement that for all $n \geq N$,

$$U(1, n) \cap [1, Cn] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, Cn].$$

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  is a (long) statement in first order logic.
- "If $u \in U(1, n) \cap [1, Cn]$, then $m_0 n + p_0 \leq u \leq k_0 n + r_0$ or..."
- Therefore, if we can prove this statement in the hyper-naturals, then it will be true in the naturals by the transfer principle.

Transfer Principle:

- For $a \in \mathbb{N}$, use an ultrafilter to extend the function

$$U(a, \cdot) : \mathbb{N} \to \mathcal{P}(\mathbb{N})$$

to a function on the hyper-naturals.

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- Note that the following first order sentences must be true of $U(a,b)$.
  - $\forall n \leq b$, $n \in U(a,b)$ if and only if $n = a$ or $n = b$.
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- This allows us to get a handle on $U(a, \cdot)$ even over the hypernaturals.

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- To make this formal, argue by induction on C and i.
- Choose the largest i such that $[m_i\omega + p_i, k_i\omega + r_i] \subset [1, (C - 1)\omega]$. We need to build up to $C\omega$.

Some Further Details:

- Since $U(1,\omega) \cap [1, C\omega]$ is hyper-finite, there exists an element $u \in U(1, C\omega)$ that is the smallest element larger than $k_i\omega + r_i$.

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- By looking through the various cases, we see that $u = m_{i+1}\omega + p_{i+1}$, where m_{i+1}, p_{i+1} are sums of previous coefficients, ± 2.

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- We thus construct m_i, p_i, k_i, r_i such that

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- These coefficients don't depend on the choice of ω at all, so the statement is true for all larger ω', which is to say the weak rigidity theorem is true!

Growth Rate of Coefficients:

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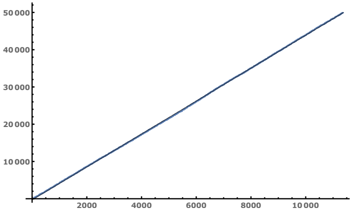
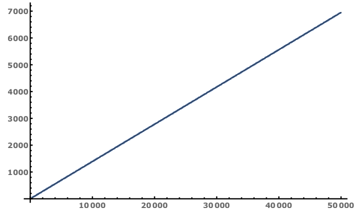
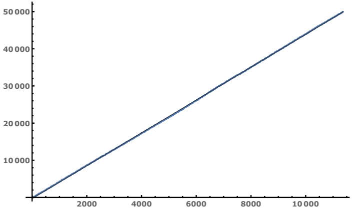


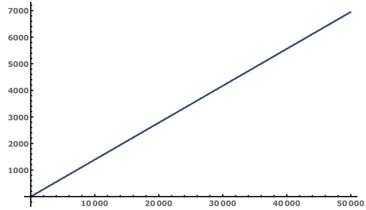
i vs. m_i



m_i vs. p_i



i vs. k_i



k_i vs. r_i

Growth Rate of Coefficients:

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$$\left|p_i - 0.139m_i\right|, \left|r_i - 0.139k_i\right| < 2.5$$

- This is useful, because we can use this statement about the growth rate to make the weak rigidity theorem effective.

Effective Estimates:

Theorem

Suppose that for some positive integer M,

$$U(1, N_0) \cap [1, k_M N_0 + r_M + 1] = \bigsqcup_{i=1}^{M} [m_i N_0 + p_i, k_i N_0 + r_i]$$

where for some $B, \epsilon > 0$, $|p_i - m_i B|, |r_i - k_i B| < \epsilon$, and $N_0 > 4(1+\epsilon) - B$. Then for all $N > N_0$,

$$U(1, N) \cap [1, k_M N + r_M + 1] = \bigsqcup_{i=1}^{M} [m_i N + p_i, k_i N + r_i].$$

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- The proof proceeds by induction over M and N.

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  1. Calculate coefficients m_i, p_i, k_i, r_i.
  2. Do a linear regression to fit the best value of B to the computed coefficients. Calculate the corresponding maximum error ϵ.
  3. Compute $N_0' = \lceil 4(1+\epsilon) - B \rceil$.
  4. Use the coefficients m_i, p_i, k_i, r_i to predict the first CN terms of $U(1, N_0'), U(1, N_0' \pm 1) \ldots$.
  5. Halt once you find the smallest N_0 such that $U(1, n)$ matches the prediction for all $N_0 \leq n \leq \max\{N_0, N_0'\}$.

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  5. Halt once you find the smallest N_0 such that $U(1, n)$ matches the prediction for all $N_0 \leq n \leq \max\{N_0, N_0'\}$.
- Using this, we prove that for all $n \geq 4$,

$$U(1, n) \cap [1, 50000n] = \bigsqcup_{i \in \mathbb{N}} [m_i n + p_i, k_i n + r_i] \cap [1, 50000n].$$

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Theorem (Weak Rigidity Theorem)

For every positive integer C, there exist integers $L, N_0 \geq 1$ such that for every congruence class $c \mod L$, if $N \geq N_0$ and $N \equiv c \mod L$, we can decompose $U(a, N)$ as a disjoint union,

$$U(a,N) \cap [1, CN] = \left(\bigsqcup_{i \in \mathbb{N}} [m_i N + p_i, k_i N + r_i] \cap (s_i + a^{l_i}\mathbb{Z}) \right) \cap [1, CN].$$

Generalizing:

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Examples:

- If $n \equiv 1 \mod 4$ sufficiently large, then

$$U(2,n) \cap [1, 5n+2] : \boxed{2}, \boxed{n, n+2, \ldots 3n}, \boxed{2n+2}, \boxed{3n+4, 3n+8, \ldots 5n+2}.$$

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- Observations that terms of $U(a, b)$ can be expressed linearly aren't entirely new.
  - Schmerl and Spiegel (1992) use a result like this for the first $(3n+11)/2$ terms of $U(2, n)$ to prove that such sequences are regular.
  - Queneau (1969) determined the first $(2n+5)$ terms.

Applications of Rigidity:

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- Schmerl and Spiegel made use of Finch's 1991 result that if an Ulam sequence contains finitely many even terms, then it is regular.
- The statement "There exists $N > 0$ such that every larger Ulam number is odd" is manifestly first-order.
- *Open Question:* can you use a combination of model theory and elementary number theory techniques to prove a statement like "For all N sufficiently large, $U(6, N)$ is regular"?

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- *Open Question:* What is the most general family of integer subsets for which you can formulate a rigidity result?

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Individual Ulam sequences are generally chaotic. Families of Ulam sequences are deeply rigid.

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- I will buy a beer for anyone who actually manages to prove something like this.